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On the Convolution of a Box Spline with a Compactly Supported Distribution: The Exponential-Polynomials in the Linear Span

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For a given box spline B and a compactly supported distribution μ, we examine in this note the convolution $B * \mu$ and the space $H(B * \mu)$ of all exponential-polynomials spanned by its integer translates. The main result here provides a necessary and sufficient condition for the equality $H(B * \mu) = H(B)$. This condition is given in terms of the distribution of the zeros of the Fourier–Laplace transform of $B * \mu$ and allows us to reduce the above equality to much simpler settings. The importance of this result is for the determination of the approximation properties of the space spanned by the integer translates of $B * \mu$. Typical examples are discussed.

1. INTRODUCTION

The basic model in multivariate splines on a uniform mesh (=multivariate splines on a regular grid) consists of a compactly supported function ϕ defined on $\mathbb{R}^s$ and the space $S(\phi)$ spanned by its integer translates. Two of the most important criteria for a favourable choice of ϕ are the linear independence of the integer translates of ϕ, and the local approximation properties of the space $H(\phi) :=$ the set of all exponential-polynomials that lie in $S(\phi)$ (here and elsewhere, an exponential-polynomial is a linear combination of finitely many products of exponentials with polynomials). The significance of this last space is due to the fact that in most circumstances the local approximation power of $H(\phi)$ can be shown, with the aid of the so-called quasi-interpolation schemes, to provide a lower bound on the approximation power corresponding to $S(\phi)$ and appropriate scaled versions of it. However, these two basic properties (the linear independence of the integer translates and the good local approximation power of the space

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$H(\phi)$) are highly competitive properties, a fact which will be illustrated later on.

In many of the practical examples of ϕ, the compactly supported function is constructed by convolving together several functions or distributions. A tentative justification for such an approach would emphasize the fact that the functions of $H(\phi)$ are determined by the distribution of the zeros of the Fourier–Laplace transform of ϕ together with the multiplicities of these zeros; such a property can be treated more efficiently when ϕ is expressed as a convolution of simple factors.

Exponential box (EB-)splines, introduced in [R1], generalize the well-known polynomial box splines [BD, BH1] and provide a wide selection of choices for the function ϕ. To introduce a typical EB-spline, let Γ be a finite multiset (to be referred later as a *defining set*) with cardinality $\#\Gamma$ consisting of elements of the form

$$\gamma = (x_\gamma, \lambda_\gamma), \tag{1.1}$$

where $x_\gamma \in \mathbb{Z}^s \backslash 0$ and $\lambda_\gamma \in \mathbb{C}$. The EB-spline corresponding to Γ, $B(\Gamma)$, is defined via its Fourier transform by

$$\hat{B}(\Gamma \mid x) := \prod_{\gamma \in \Gamma} \hat{B}(\gamma \mid x) := \prod_{\gamma \in \Gamma} \left(\int_0^1 e^{(\lambda_\gamma - i x_\gamma \cdot x) t}\, dt \right). \tag{1.2}$$

Note that indeed the exponential box spline can be expressed as a convolution of lower order ones. In fact, if $\Gamma = \Gamma_1 \cup \Gamma_2$, it follows from (1.2) that

$$B(\Gamma) = B(\Gamma_1) * B(\Gamma_2). \tag{1.3}$$

In case

$$\langle \Gamma \rangle := \operatorname{span}\{x_\gamma\}_{\gamma \in \Gamma} = \mathbb{R}^s, \tag{1.4}$$

$B(\Gamma)$ gives rise to a compactly supported function $B(\Gamma \mid \cdot)$; otherwise the EB-spline is merely a distribution (actually a measure) supported in $\langle \Gamma \rangle$. For more information about EB-splines we refer the reader to [R1, R2, BR, DM, DR].

Only a few other examples of a function ϕ can be found in the literature, and most of these examples consist of bivariate piecewise-polynomials. In fact some of these functions are obtained by convolving a (polynomial) box spline with a certain (and simple) compactly supported function (cf., e.g., [BH2, CH]). Stimulated by the latter functions, we became interested in the properties of a function ψ obtained as a convolution of an EB-spline and an arbitrary compactly supported distribution μ. For that model, the question of linear independence of the integer translates has been discussed thoroughly in [CR]. In this note we compare $H(\psi)$ (with ψ as above) with $H(B(\Gamma))$. Our main result here provides a necessary and sufficient condition for the equality

$$H(\psi) = H(B(\Gamma)). \tag{1.5}$$

The statement as well as the proof of the main result is presented in Section 3. In Section 2 we collect the notations and preliminaries needed for this proof. Finally, we discuss in Section 4 two examples which demonstrate the efficiency of the main result.

This paper together with [CR] allows us to determine clear criteria for a "good" choice of μ. The various applications of these results will be studied in a subsequent paper of C. K. Chui and the author.

2. Notations and Preliminaries

Throughout this paper we use Π for the space of all s-dimensional polynomials, e_θ for the exponential $\exp(i\theta\cdot)$, and $\hat{\phi}$ for the Fourier–Laplace transform of the compactly supported distribution ϕ, i.e., the analytic continuation of the Fourier transform of ϕ. $E^x, x \in \mathbb{R}^s$, stands for the translation operation

$$E^x : f \mapsto (\cdot + x),$$

and the terminology *difference operator* is used exclusively for finite linear combinations of integral translations.

Given a subset K of the defining set Γ, we find it convenient to refer to linear properties of $\{x_\gamma\}_{\gamma \in K}$ in terms of K. Thus, we say that K is *linearly-independent* and mean that the vectors $\{x_\gamma\}_{\gamma \in K}$ are linearly independent. Also we use

$$\langle K \rangle$$

for the *real* span of $\{x_\gamma\}_{\gamma \in K}$, and

$$K^\perp$$

for the *complex* set

$$\{x \in \mathbb{C}^s \mid x \cdot x_\gamma = 0, \forall \gamma \in K\}.$$

For $x \in \mathbb{R}^s \backslash 0$, let D_x be the directional derivative in the x-direction. Given $K \subset \Gamma$, we set

$$D^K := \prod_{\gamma \in K} D^\gamma := \prod_{\gamma \in K} (D_{x_\gamma} - \lambda_\gamma).$$

The differential operators of the form D^K play an important role in box spline theory; particularly we have (cf. [R1, Theorem 2.2])

(2.1) PROPOSITION. *For $K \subset \Gamma$*

$$D^K B(\Gamma) = \nabla^K B(\Gamma \backslash K),$$

where ∇^K is the difference operator

$$\nabla^K = \prod_{\gamma \in K} (E^0 - e^{\lambda_\gamma} E^{-x_\gamma}).$$

The above differential operators are also important in the analysis of $H(B(\Gamma))$. To discuss this part, we first introduce the following collection of subsets of Γ,

$$\mathbb{K}(\Gamma) := \{K \subset \Gamma \mid \langle \Gamma \backslash K \rangle \neq \mathbb{R}^s\},$$

and associate the defining set Γ with the space

$$\mathscr{H}(\Gamma) := \{f \in \mathscr{D}'(\mathbb{R}^s) : D^K f = 0, \forall K \in \mathbb{K}(\Gamma)\}.$$

It is known (cf. Section 4 of [BR]) that $\mathscr{H}(\Gamma)$ is an exponential-polynomial space (namely, is spanned by products of exponentials with polynomials). The significance of $\mathscr{H}(\Gamma)$ in our context lies in the fact that $B(\Gamma)$ is a piecewise-$\mathscr{H}(\Gamma)$ function, (in particular $H(B(\Gamma)) \subset \mathscr{H}(\Gamma)$), and that generically $H(B(\Gamma)) = \mathscr{H}(\Gamma)$. More precisely, we have (cf., e.g., [BR, Theorem 6.2]):

(2.2) PROPOSITION. *If, for some $\theta \in \mathbb{C}^s$, $\hat{B}(\Gamma \mid \theta) \neq 0$, then*

$$e_\theta \Pi \cap \mathscr{H}(\Gamma) = e_\theta \Pi \cap H(B(\Gamma)).$$

We now discuss the Fourier analysis elements which are needed in the sequel. First note that (1.2) implies that

$$\hat{B}(\gamma \mid x) = 0 \Leftrightarrow \lambda_\gamma - ix \cdot x_\gamma \in 2\pi i \mathbb{Z} \backslash 0. \tag{2.3}$$

The next result (which, essentially, is known [B]) provides, for an arbitrary compactly supported distribution, a characterization of $H(\phi)$ in terms of the distribution of the zeros of the Fourier transform of ϕ. We use the notation $\phi *' f$ for the semi-discrete convolution, i.e.,

$$\phi *' f := \sum_{\alpha \in \mathbb{Z}^s} f(\alpha)\, \phi(\cdot - \alpha), \tag{2.4}$$

where f is defined (at least) on $\mathbb{Z}^s$.

(2.5) THEOREM. *Let ϕ be a compactly supported distribution and $\theta \in \mathbb{C}^s$. If, for some $p \in \Pi$, $e_\theta p \in H(\phi)$, then*

$$(E^\beta p)(-iD)\, \hat{\phi}(\theta + 2\pi\alpha) = 0, \qquad \forall \alpha \in \mathbb{Z}^s \backslash 0,\ \beta \in \mathbb{Z}^s. \tag{2.6}$$

Moreover, if, in addition, $\hat{\phi}(\theta) \neq 0$, then the converse implication holds as well.

Proof. We first note that Theorem 2.7 of [BoR2] shows that condition (2.6) is equivalent to the condition

$$\phi *' e_\theta p \in e_\theta \Pi. \tag{2.7}$$

Further, if we assume that $\hat{\phi}(\theta) \neq 0$, then it is known that

$$e_\theta p \in H(\phi) \Leftrightarrow \phi *' e_\theta p \in e_\theta \Pi$$

(Proposition 2.2 of [RS] proves the case $\theta = 0$, to which the general case is a straightforward extension), and the equivalence between (2.6) and the condition $e_\theta p \in H(\phi)$ thus follows.

It remains to show that the assumption $e_\theta p \in H(\phi)$ implies (2.7), without any reliance on the condition $\hat{\phi}(\theta) \neq 0$: assuming $e_\theta p \in H(\phi)$, we obtain from Corollary 5.5 of [BoR1] that $e_\theta p = \phi *' e_\theta r$, for some $r \in \Pi$, hence, since $\phi *'$ commutes with integer translations, we even have $\nabla(e_\theta p) = \phi *' \nabla(e_\theta r)$, for every difference operator ∇. On the other hand, invoking the implication (c) $\Rightarrow$ (b) of Theorem 2.7 of [BoR2], we see that also $e_\theta p = \phi * e_\theta r$, and consequently $e_\theta p$ may be written in the form $\nabla(e_\theta r)$ for a suitably chosen ∇, since any convolution operator on a translation-invariant finite-dimensional subspace of $e_\theta \Pi$ can be represented as a difference operator. Therefore, for any such ∇, $\nabla(e_\theta p) = \phi *' e_\theta p$, and thus (2.7) (hence also (2.6)) holds. ∎

The special case $\deg p = 0$ in Theorem 2.5 will be used frequently in the sequel, hence is stated separately:

(2.8) COROLLARY. *Let ϕ be a compactly supported distribution and $\theta \in \mathbb{C}^s$. If $e_\theta \in H(\phi)$, then*

$$\hat{\phi}(\theta + 2\pi\alpha) = 0, \qquad \forall \alpha \in \mathbb{Z}^s \backslash 0, \tag{2.9}$$

and the converse is true if $\hat{\phi}(\theta) \neq 0$.

If (2.9) holds, we obtain

$$\phi *' e_\theta = \hat{\phi}(\theta)\, e_\theta, \tag{2.10}$$

which implies [R3] that the condition

$$\hat{\phi}(\theta + 2\pi\alpha) = 0, \qquad \forall \alpha \in \mathbb{Z}^s$$

is sufficient for the linear dependence of the integer translates of ϕ. A comparison of this last condition with (2.9) demonstrates the competition between the properties of a rich $H(\phi)$ on the one hand, and linear independence of the integer translates of ϕ on the other hand.

3. THE MAIN RESULT

Throughout this section μ is a fixed compactly supported distribution and $B(\Gamma)$ denotes an exponential box spline whose defining set Γ satisfies

$$\langle \Gamma \rangle = \mathbb{R}^s. \tag{3.1}$$

Given a subset $M \subset \Gamma$ we set

$$\psi_M := B(M) * \mu. \tag{3.2}$$

The following theorem is the key for the desired necessary and sufficient condition for the equality

$$H(B(\Gamma)) = H(\psi_\Gamma). \tag{3.3}$$

(3.4) THEOREM. *Let θ be in $\mathbb{C}^s$, and assume that $\hat{\psi}_\Gamma(\theta) \neq 0$. Then the following conditions are equivalent*:

(a) *For some $p \in \Pi$,*

$$e_\theta\, p \in H(\psi_\Gamma) \backslash \mathscr{H}(\Gamma). \tag{3.5}$$

(b) *There exists some linearly independent subset $M \subset \Gamma$ of cardinality $< s$ such that*

$$e_\theta \in H(\psi_M). \tag{3.6}$$

(c) *There exists some linearly independent subset $M \subset \Gamma$ of cardinality $<s$ such that*

$$\lambda_\gamma - i\theta \cdot x_\gamma = 0, \qquad \forall \gamma \in M, \tag{3.7}$$

and

$$\hat{\mu}(\theta + 2\pi\alpha) = 0, \qquad \forall \alpha \in \mathbb{Z}^s \cap M^\perp \backslash 0. \tag{3.8}$$

Proof of Theorem 3.4. We start the proof by showing that (b)$\Leftrightarrow$(c). This equivalence is the content of the following two claims.

CLAIM 1. *Let M be any subset of Γ that satisfies* (3.7) *and* (3.8). *Then* (3.6) *holds with respect to this M.*

Proof. Since we assume $\hat{\psi}_\Gamma(\theta) \neq 0$, it follows that $\hat{\psi}_M(\theta) \neq 0$ (since $\psi_\Gamma = \psi_M * B(\Gamma \backslash M)$). Therefore, in view of Corollary 2.8, the claim will be proved as soon as we show that

$$\hat{\psi}_M(\theta + 2\pi\alpha) = \hat{B}(M \mid \theta + 2\pi\alpha)\, \hat{\mu}(\theta + 2\pi\alpha) = 0, \qquad \forall \alpha \in \mathbb{Z}^s \backslash 0. \tag{3.9}$$

For $\alpha \in \mathbb{Z}^s \cap M^\perp \backslash 0$, (3.9) is guaranteed by (3.8). Otherwise, there exists $\gamma \in M$ such that $\alpha \cdot x_\gamma \neq 0$; this means, in view of (3.7) (and since $x_\gamma \in \mathbb{Z}^s$), that

$$\lambda_\gamma - i(\theta + 2\pi\alpha) \cdot x_\gamma = (\lambda_\gamma - i\theta \cdot x_\gamma) - i2\pi\alpha \cdot x_\gamma \in 2\pi i\mathbb{Z} \backslash 0, \tag{3.10}$$

and thus, by (2.3), $\hat{B}(\gamma \mid \theta + 2\pi\alpha) = 0$. We conclude that

$$\hat{B}(M \mid \theta + 2\pi\alpha) = 0, \qquad \forall \alpha \in \mathbb{Z}^s \backslash M^\perp, \tag{3.11}$$

and hence (3.9) is verified and the claim is thus proved.

CLAIM 2. *Assume that for some $M \subset \Gamma$, $e_\theta \in H(\psi_M)$, and that M is minimal with respect to this property. Then M is necessarily linearly independent and satisfies* (3.7), (3.8).

Proof. Since, by assumption, $\hat{\psi}_M(\theta) \neq 0$, Corollary 2.8 allows us to conclude that (3.9) is equivalent to the assumption that $e_\theta \in H(\psi_M)$. Therefore, the minimality of M implies the existence of $\{\alpha_\gamma\}_{\gamma \in M} \subset \mathbb{Z}^s \backslash 0$ such that

$$\hat{B}(\gamma \mid \theta + 2\pi\alpha_\gamma) = 0, \qquad \gamma \in M. \tag{3.12}$$

(Indeed, since we assume $e_\theta \in H(\psi_M)$, then, by Corollary 2.8, $\hat{\psi}_M$ vanishes on $(\theta + 2\pi\mathbb{Z}^s) \backslash 0$, and if, for some $\gamma \in M$, $\hat{B}_\gamma$ vanishes *nowhere* on

$(\theta+2\pi\mathbb{Z}^s)\backslash 0$, then also $\hat{\psi}_{M\backslash\gamma}$ would vanish on $(\theta+2\pi\mathbb{Z}^s)\backslash 0$, hence (Corollary 2.8) $e_\theta \in H(\psi_{M\backslash\gamma})$, contradicting thereby the minimality of M.) From (3.12) and (2.3) we obtain that

$$\lambda_\gamma - i(\theta+2\pi\alpha)\cdot x_\gamma \in 2\pi i\mathbb{Z}, \qquad \forall\alpha\in\mathbb{Z}^s, \gamma\in M; \tag{3.13}$$

in particular

$$\lambda_\gamma - i\theta\cdot x_\gamma \in 2\pi i\mathbb{Z}, \qquad \forall\gamma\in M. \tag{3.14}$$

Utilizing the fact that the assumption $\hat{\psi}_\Gamma(\theta)\neq 0$ implies $\hat{B}(\gamma\,|\,\theta)\neq 0$ for all $\gamma\in\Gamma$, we may combine (3.14) together with (2.3) to conclude

$$\lambda_\gamma - i\theta\cdot x_\gamma = 0, \qquad \forall\gamma\in M, \tag{3.15}$$

that is, (3.7) holds.

Now, let $\alpha\in M^\perp\backslash 0$; then (3.15) shows that

$$\lambda_\gamma - i(\theta+2\pi\alpha)\cdot x_\gamma = 0, \qquad \forall\gamma\in M,$$

which yields (in view of (2.3)) that for such an α

$$\hat{B}(M\,|\,\theta+2\pi\alpha)\neq 0,$$

and thus (3.9) forces

$$\hat{\mu}(\theta+2\pi\alpha)=0,$$

which proves (3.8).

To complete the proof of the claim, it remains to show that M is necessarily linearly independent. Let $M_1\subset M$ be a linearly independent set that spans $\langle M\rangle$. Since (3.7) and (3.8) hold with respect to M, they hold with respect to M_1, and therefore Claim 1 implies that (3.6) is valid with M_1 replacing M. The minimality of M then ensures that $M_1=M$, so M is indeed linearly independent and Claim 2 is thus established.

We now prove the implication (a) $\Rightarrow$ (b). For that we need the following fact of independent interest:

(3.16) PROPOSITION. *In the notations of the theorem, if $e_\theta p\in H(\psi_\Gamma)\backslash \mathscr{H}(\Gamma)$, then (without requiring the assumption $\hat{\psi}_\Gamma(\theta)\neq 0$) there exists $M\subset\Gamma$, such that $e_\theta\in H(\psi_M)$, while $\langle M\rangle\neq\mathbb{R}^s$.*

Proof. Since $e_\theta p\in H(\psi_\Gamma)$, there exists a sequence c such that $\psi_\Gamma *' c = e_\theta p$. On the other hand, since $e_\theta p\notin\mathscr{H}(\Gamma)$, there exists some $K\in\mathbb{K}(\Gamma)$ such that $D^K(e_\theta p)\neq 0$. Since $D^K(e_\theta p)\in e_\theta\Pi$, we obtain, by Proposition 2.1,

$$e_\theta\Pi\backslash 0 \ni D^K(e_\theta p) = D^K(B(\Gamma)*\mu *' c) = B(\Gamma\backslash K)*\mu *'\nabla^K c. \tag{3.17}$$

This shows that, with $M := \Gamma\backslash K$, $H(\psi_M) \cap e_\theta \Pi \neq 0$. Further, the space $H(\psi_M) \cap e_\theta \Pi$ is invariant under integer translates, hence [B] invariant under any translates. Since every non-trivial translation-invariant subspace of $e_\theta \Pi$ must contain e_θ, we conclude that $e_\theta \in H(\psi_M)$, while by the definition of $\mathbb{K}(\Gamma)$, $\langle M \rangle \neq \mathbb{R}^s$.

The implication (a) $\Rightarrow$ (b) of the theorem readily follows from Proposition 3.16, since we need only show that M in the proposition can be replaced by a linearly independent subset of it, which is simple: if M_1 is a minimal subset of M with respect to the property $e_\theta \in H(\psi_{M_1})$, then, by Claim 2, M_1 is necessarily linearly independent.

It remains to prove that (c) implies (a). Here, let M be the set appearing in (c) and let ξ be any non-trivial vector in $M^\perp$. Define

$$p(x) := (\xi \cdot x)^k, \tag{3.18}$$

where k is the least non-negative integer satisfying

$$f := e_\theta p \notin \mathscr{H}(\Gamma).$$

We contend that f satisfies (a), i.e., $f \in H(\psi_\Gamma)$.

Let us show that indeed $f \in H(\psi_\Gamma)$. Since we assume $\hat{\psi}_\Gamma(\theta) \neq 0$, application of Theorem 2.5 yields that this will be established as soon as we prove that

$$(D_\xi)^j \hat{\psi}_\Gamma(\theta + 2\pi\alpha) = 0, \qquad \forall \alpha \in \mathbb{Z}^s\backslash 0,\ j = 0, ..., k. \tag{3.19}$$

In the verification of (3.19) we consider two types of points:

(1) $\alpha \in \mathbb{Z}^s\backslash M^\perp$: for such an α choose $\gamma \in M$ such that $\alpha \cdot x_\gamma \neq 0$. By appealing to (3.7) we obtain

$$\lambda_\gamma - i(\theta + 2\pi\alpha) \cdot x_\gamma \in 2\pi i \mathbb{Z}\backslash 0,$$

hence the Fourier transform of $B(\gamma)$ vanishes at $\theta + 2\pi\alpha$. On the other hand, by (1.2), this transform is constant along any direction orthogonal to x_γ. Since $\gamma \in M$ and $\xi \in M^\perp$, it follows that $x_\gamma \perp \xi$. We conclude that $\hat{B}(\gamma)$, and hence $\hat{\psi}_\Gamma$, vanishes on the line

$$\{(\theta + 2\pi\alpha + t\xi)\}_{t \in \mathbb{R}}. \tag{3.20}$$

Now it is clear that for such an α, (3.19) holds (even without any restriction on j).

(2) Let $\alpha \in \mathbb{Z}^s \cap M^\perp\backslash 0$. Since $\hat{\psi}_\Gamma(\theta) \neq 0$, then also $\hat{B}(\Gamma \mid \theta) \neq 0$, and therefore, by Proposition 2.2, the assumption $e_\theta(x)(\xi \cdot x)^{k-1} \in \mathscr{H}(\Gamma)$

implies that $e_\theta(x)(\xi \cdot x)^{k-1} \in H(B(\Gamma))$. Consequently, an application of Theorem 2.5 yields that

$$(D_\xi)^j \hat{B}(\Gamma \mid \theta + 2\pi\alpha) = 0, \qquad \forall \alpha \in \mathbb{Z}^s \backslash 0,\ j = 0, ..., k-1,$$

which means that $\hat{B}(\Gamma \mid \theta + 2\pi\alpha + \cdot\, \xi)$ has a k-fold zero at 0. Since by (3.8), $\hat{\mu}(\theta + 2\pi\alpha) = 0$ as well, we conclude that $\hat{\psi}_\Gamma(\theta + 2\pi\alpha + \cdot\, \xi)$ has a $(k+1)$-fold zero at 0, and (3.19) thus holds for this case as well.

The implication (c) $\Rightarrow$ (a) is now established, and the proof of the theorem therefore comes to its end. ∎

The equivalence of (a) and (b) in the above theorem leads to the following result:

(3.21) COROLLARY. *In the notations of the previous theorem, if we assume that*

$$H(\psi_\Gamma) = \sum_{\theta \in \Theta \subset \mathbb{C}^s} e_\theta Q_\theta, \qquad Q_\theta \subset \Pi, \tag{3.22}$$

and that, for every $M \subset \Gamma$ *and* $\theta \in \mathbb{C}^s$,

$$e_\theta \in H(\psi_M) \Rightarrow \hat{\psi}_\Gamma(\theta) \neq 0, \tag{3.23}$$

then the following conditions are equivalent:

(a) $H(\psi_\Gamma) \backslash H(B(\Gamma)) \neq \varnothing$.

(b) *For some linearly independent set* $M \subset \Gamma$ *of cardinality* $< s$, $H(\psi_M)$ *contains an exponential* e_θ.

Proof. Assume (a). By (3.22), there exists an exponential-polynomial $e_\theta p \in H(\psi_\Gamma) \backslash H(B(\Gamma))$. As previously noted, this necessarily implies that $e_\theta \in H(\psi_\Gamma)$, which yields, in view of (3.23), that $\hat{\psi}_\Gamma(\theta) \neq 0$, hence also $\hat{B}(\Gamma \mid \theta) \neq 0$, and thus, by Proposition 2.2, $e_\theta p \notin \mathscr{H}(\Gamma)$ (since $e_\theta p \notin H(B(\Gamma))$). We have thus shown that $e_\theta p \in H(\psi_\Gamma) \backslash \mathscr{H}(\Gamma)$, and (b) here is obtained from the implication (a) $\Rightarrow$ (b) in Theorem 3.4.

Now, assume (b). By (3.23), since $e_\theta \in H(\psi_M)$, $\hat{\psi}_\Gamma(\theta) \neq 0$, and we may invoke the implication (b) $\Rightarrow$ (a) of Theorem 3.4 to conclude that for some $p \in H$, $e_\theta p \in H(\psi_\Gamma) \backslash \mathscr{H}(\Gamma)$, a fortiori $e_\theta p \in H(\psi_\Gamma) \backslash H(B(\Gamma))$. ∎

We note that condition (3.22) in the last corollary is a mild one. It is satisfied, e.g., whenever, for the Θ there, $2\pi\mathbb{Z}^s \cap (\Theta - \Theta) = 0$, as follows from Lemma 2.4 of [BoR2].

Simpler conditions are obtained if we assume that $B(\Gamma)$ is a *polynomial* box spline (i.e., all λ_γ's are zero), and if, subsequently, we restrict our attention to the polynomials in $H(B(\Gamma) * \mu)$. In such a case (as verified from (1.2)), $\hat{B}(\Gamma|0)\neq 0$, hence the case $\theta=0$ in Theorem 3.4 implies the following.

(3.24) COROLLARY. *Assume that $B(\Gamma)$ is a polynomial box spline, and μ is a compactly supported distribution that satisfies $\hat{\mu}(0)\neq 0$. Then $H(B(\Gamma)*\mu)\backslash H(B(\Gamma))$ contains polynomials, if and only if $1\in H(B(M)*\mu)$ for some linearly independent M of cardinality $<s$.*

4. EXAMPLES

We discuss two bivariate examples, which illustrate the results of the previous section.

(4.1) EXAMPLE. Let $B(\Gamma)$ be a bivariate three-directional exponential box spline, that is,

$$x_\gamma \in \{(1,0),(0,1),(1,1)\}, \qquad \forall\gamma.$$

Let $\mu\in L_\infty(\mathbb{R}^2)$ be supported in the triangle with vertices at (0, 0), (1, 0), (1, 1). Define, as before, $\psi := B(\Gamma)*\mu$. (Certain smooth piecewise-polynomials of minimal support are obtained in this way; cf. [CH] and the references therein.) We contend that

$$H(\psi)\subset \mathscr{H}(\Gamma), \tag{4.2}$$

which, roughly speaking, means that the approximation properties of $B(\Gamma)$ are not improved in the smoothing process $B(\Gamma)\mapsto B(\Gamma)*\mu$.

To prove (4.2), we make use of Proposition 3.16. Assuming (4.2) is not valid (for the sake of contradiction), this proposition implies the existence of $\theta\in\mathbb{C}^2$ and a subset $M\subset\Gamma$, with $\langle M\rangle\neq\mathbb{R}^2$, such that $e_\theta\in H(B(M)*\mu)$, and hence, for some sequence c, $\mu*(B(M)*' c)=e_\theta$. This is clearly impossible if $M=\varnothing$, since the supports of the integer translates of μ do not fill all of $\mathbb{R}^2$; hence we must have $\langle M\rangle=\mathbb{R}$, a case in which $B(M)$ is a measure supported on $\langle M\rangle$, which can be identified (when identifying $\langle M\rangle$ with $\mathbb{R}$) with a suitable univariate B-spline. This shows that $B(M)*' c$ is a measure supported on the union of lines

$$\bigcup_{\alpha\in\mathbb{Z}^2} \alpha+\langle M\rangle, \tag{4.3}$$

and which can be identified on each line with some locally bounded function. Furthermore, by the assumptions on μ and Γ, one can choose $\{x_h\}_{h>0}\subset\mathbb{R}^2$ such that the intersection of $x_h+\operatorname{supp}\mu$ with the set in (4.3) is a segment of length $O(h)$. It follows that $\mu * (B(M) *' c)$ is either discontinuous or has some zeros, hence cannot be the exponential e_θ. Thus, we obtained the desired contradiction, and (4.2) follows.

(4.4) EXAMPLE. Here, we assume that $B(\Gamma)$ is a bivariate polynomial box spline, and μ is the characteristic function of some domain Ω. In such a case, $H(B(\Gamma)) = \mathscr{H}(\Gamma) \subset \Pi$. Suppose that we want to choose μ such that $H(B(\Gamma) * \mu) \backslash H(B(\Gamma))$ contains polynomials. How to do that? To answer this question we invoke Corollary 3.24 (since μ is the characteristic function of some domain, $\hat{\mu}(0) \neq 0$, so that this corollary is applicable) to conclude that the condition needed is that, for some $\gamma \in \Gamma$, $1 \in H(B(\gamma) * \mu)$, or equivalently [B],

$$B(\gamma) * \mu *' 1 \in \Pi.$$

(As a matter of fact, M might be of cardinality either 1 or 0; yet the latter case is the trivial situation when $1 \in H(\mu)$.) Since $B(\gamma)$ is a measure whose mass is uniformly distributed on the segment $\{tx_\gamma\}_{0 \leq t \leq 1}$ both of whose endpoints are integral, it follows that $B(\gamma) *' 1$ is a measure whose mass is uniformly distributed on the lines

$$\{\alpha + \operatorname{span} x_\gamma\}_{\alpha \in \mathbb{Z}^2}. \tag{4.5}$$

Thus, $\mu * (B(\gamma) *' 1)(x)$ is the sum of the lengths of the intersection segments of the lines in (4.5) with the x-translate of μ. Consequently, in order for $H(B(\Gamma) * \mu)$ to contain polynomials not already in $H(B(\Gamma))$ it is necessary and sufficient that, in one of the directions $\{x_\gamma\}_{\gamma \in \Gamma}$, the $\mathbb{R}$-Lebesgue measure of the sets

$$\{\alpha + tx_\gamma + x : \alpha \in \mathbb{Z}^2, t \in \mathbb{R}\}, \qquad x \in \mathbb{R}^2$$

be independent of x a.e. ($\mathbb{R}^2$).

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